Type **ODE SERIES SOLUTION** [redacted]

(2) Solve the equation

$$\frac{d^2y}{dx^2} + (x-1)^2\frac{dy}{dx} - 4(x-1)y = 0 \quad \text{——— (1)}$$

in series about the ordinary point $x = 1$

Sol^n Let us take the transformation

$z = x - 1$ i.e $x - 1 = z \longrightarrow \frac{dz}{dx} = 1$

$$\therefore \frac{dx}{dy} = \frac{dz}{dy}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dy}{dz}} \longrightarrow \frac{d}{dx}(y) = \frac{d}{dz}(y)$$

$$\Rightarrow \boxed{\frac{d}{dx} = \frac{d}{dz}} \text{ (i.e operator for same)}$$

Also,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$= \frac{d^2y}{dz^2}\cdot 1$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}$$

Using transformation
This is the process-1,
follow next for process-2

putting these in (1) we get

$$\frac{d^2y}{dz^2} + z^2\cdot\frac{dy}{dz} - 4zy = 0 \quad \text{——— (2)}$$

which shows that $z = 0$ is an ordinary point of the equation (2)

Let us assume the solution of (2) as

$$y = \sum_{n=0}^{\infty} a_n z^n \quad \text{——— (3)}$$

Differentiating (3) w.r.t. z twice, we get

$$\frac{dy}{dz} = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

$$\frac{d^2y}{dz^2} = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$$

Substituting these in (2) we get

$$\sum_{n=2} n(n-1)a_n z^{n-2} + z^2\sum_{n=1}^{\infty} n a_n z^{n-1} - 4z\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=1}^{\infty} n a_n z^{n+1} - 4\sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n + \sum_{n=2}^{\infty}(n-1)a_{n-1} z^n - 4\sum_{n=1}^{\infty} a_{n-1} z^n = 0$$

$$\Rightarrow \sum_{n=0}^{1}(n+2)(n+1)a_{n+2}z^n + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}z^n$$
$$+ \sum_{n=2}^{\infty}(n-1)a_{n-1}z^n - 4a_0 z - 4\sum_{n=2}^{\infty} a_{n-1}z^n = 0$$

$$\Rightarrow 2a_2 + 6a_3 z + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}z^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}z^n$$
$$-4a_0 z - 4\sum_{n=2}^{\infty} a_{n-1}z^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} - 4a_{n-1}\right]z^n$$
$$+ 2a_2 + (6a_3 - 4a_0)z = 0 \quad \text{———} \quad (4)$$

Relation (4) is valid $\forall z\ (-R, R)$ where R is the radius of convergence of the power series (3)

Therefore the coefficient of each term in the L.H.S of (4) must be zero, which gives

Because it is an identity

$$2a_2 = 0 \Rightarrow a_2 = 0$$

$$6a_3 - 4a_0 \Rightarrow a_3 = \frac{2a_0}{3}$$

$$(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} - 4a_{n-1} = 0$$

$$\Rightarrow (n+2)(n+1)a_{n+2} + n a_{n-1} - 5a_{n-1} = 0$$

$$\Rightarrow a_{n+2} = \frac{5a_{n-1} - n a_{n-1}}{(n+2)(n+1)}, \quad \forall n \geq 2 \quad \text{[Recurrence Relation.]} \quad \text{——— (5)}$$

putting, $n = 2, 3, 4$ in (5) respectively, we get

$$a_4 = \frac{5a_1 - 2a_1}{12} = \frac{a_1}{4}$$

$$a_5 = \frac{5a_2 - 3a_2}{20} = \frac{2a_2}{20} = 0 \quad [\because a_2 = 0]$$

$$a_6 = \frac{5a_3 - 4a_3}{30} = \frac{a_3}{30} = \frac{2a_0}{90} = \frac{a_0}{45}$$

.... and so on.

Now, substituting all the values of $a_2, a_3, a_4, a_5, a_6, \ldots\ldots$ in (3) we get

$$y = \sum_{n=0}^{\infty} a_n z^n$$

$$= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 + \ldots\ldots$$

$$= a_0 + a_1 z + \frac{2}{3} a_0 z^3 + \frac{a_1}{4} z^4 + \frac{1}{45} a_0 z^6 + \ldots\ldots$$

$$= a_0 \left(1 + \frac{2}{3} z^3 + \frac{1}{45} z^6 + \ldots\ldots\right) + a_1 \left(z + \frac{1}{4} z^4 + \ldots\ldots\right)$$

$$= a_0 \left\{1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 + \ldots\right\} + a_1 \left\{(x-1) + \frac{1}{4}(x-1)^4 + \ldots\right\}$$

where a_0, a_1 are arbitrary constant.
which is the required solution.

$\boxed{\text{process} - 2}$

Find the general solution of the ODE

$$\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} + y = 0 \quad \text{near } 2.$$

sol^n.

$$\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} + y = 0 \quad \text{------------} \quad (1)$$

Here, $P(x) = (x-1)$

$Q(x) = 1$

– which are both analytic at $x = 2$

$\therefore x = 2$ is an ordinary point.

Let us assume a solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n \quad \text{------------} \quad (2)$$

Differentiating (2) w.r.t. x twice we get

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2}$$

Substituting these in (1) we get

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} + (x-1)\sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} + (x-2+1)\sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n (x-2)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-2)^n + \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-2)^n + \sum_{n=1}^{\infty} n a_n (x-2)^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-2)^n + \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-2)^n + \sum_{n=1}^{\infty} n a_n (x-2)^n + a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} (x-2)^n + a_0 + \sum_{n=1}^{\infty} a_n (x-2)^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + (n+1) a_{n+1} + a_n \right] (x-2)^n + (2a_2 + a_1 + a_0) = 0 \quad \text{———} (3)$$

Relation (3) is valid $\forall (x-2) \in (-R, R)$, where R is the radius of convergence of the power series (2).

Therefore the coefficients of each term in the L.H.S of (3) must be zero, which gives

$$2a_2 + a_1 + a_0 = 0 \Rightarrow a_2 = -\frac{1}{2}a_1 - \frac{1}{2}a_0$$

$$(n+2)(n+1) a_{n+2} + n a_n + (n+1) a_{n+1} + a_n = 0, \quad \forall n \geq 1$$

$$\Rightarrow a_{n+2} = \frac{-(n+1)a_n - (n+1)a_{n+1}}{(n+2)(n+1)}, \quad \forall n \geq 1 \quad \text{———} (4)$$

putting $n = 1, 2$ in (4) respectively, we get

$$a_3 = \frac{-2a_1 - 2a_2}{6} = -\frac{1}{3}a_1 - \frac{1}{3}a_2$$

$$= -\frac{1}{3}a_1 - \frac{1}{3}\left\{-\frac{1}{2}a_1 - \frac{1}{2}a_0\right\}$$

$$= -\frac{1}{3}a_1 + \frac{1}{6}a_1 + \frac{1}{6}a_0$$

$$= -\frac{a_1}{6} + \frac{a_0}{6}$$

$$a_4 = \frac{-3a_2 - 3a_3}{12} = -\frac{1}{4}a_2 - \frac{1}{4}a_3$$

$$= -\frac{1}{4}\left\{-\frac{1}{2}a_1 - \frac{1}{2}a_0\right\} - \frac{1}{4}\left\{-\frac{a_1}{6} + \frac{a_0}{6}\right\}$$

$$= \frac{1}{8}a_1 + \frac{1}{8}a_0 + \frac{1}{24}a_1 - \frac{1}{24}a_0$$

$$= \frac{1}{6}a_1 + \frac{1}{12}a_0$$

------- and so on.

Now, substituting all the values of $a_2, a_3, a_4, \ldots$ in (2) we get

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \cdots$$

$$= a_0 + a_1(x-2) - \frac{1}{2}a_1(x-2)^2 - \frac{1}{2}a_0(x-2)^2 - \frac{a_1}{6}(x-2)^3 + \frac{a_0}{6}(x-2)^3 + \frac{a_1}{6}(x-2)^4 + \frac{1}{12}a_0(x-2)^4 + \cdots$$

$$= a_0\left\{1 - \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{12}(x-2)^4 + \cdots\right\} + a_1\left\{(x-2) - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 + \cdots\right\}$$

where a_0, a_1 are arbitrary constants.

which is the required solution.