[redacted]



**(A) Locate and classify the singular points (if exist) :-**

**(1)** $x(x-1)^3 \dfrac{d^2y}{dx^2} - 2(x-1)^2 y' + 5y = 0$

Its normalised formed is

$$y'' - \frac{2}{x(x-1)} y' + \frac{5}{x(x-1)^3} y = 0$$

Here, $P(x) = -\dfrac{2}{x(x-1)}$, is not analytic at $x=0$ and $x=1$

$Q(x) = \dfrac{5}{x(x-1)^3}$, is not analytic at $x=0$ and $x=1$

$\therefore$ $x = 0, 1$ are the singular points.

Now, $(x-0) \cdot P(x) = -\dfrac{2x}{x(x-1)} = -\dfrac{2}{x-1}$, is analytic at $x=0$

$(x-0)^2 \cdot Q(x) = \dfrac{5x}{(x-1)^3}$, is analytic at $x=0$

$\therefore$ $x=0$ is a regular singular point.

Again, $(x-1)P(x) = -\dfrac{2}{x}$, is analytic at $x=1$

$(x-1)^2 Q(x) = \dfrac{5}{x(x-1)}$, is not analytic at $x=1$.

$\therefore$ $x=1$ is an irregular singular point.

(2) $(x^2-3x)y''+(x+1)y'-2y=0$

Its normalised form is

$$y''+\frac{x+1}{x(x-3)}y'-\frac{2}{x(x-3)}y=0$$

Here, $P(x)=\dfrac{x+1}{x(x-3)}$, is not analytic at $x=0$ and $x=3$

$Q(x)=-\dfrac{2}{x(x-3)}$, is not analytic at $x=0$, $x=+3$

$\therefore$ $x=0,3$ are the singular points.

Now, $(x-0)P(x)=\dfrac{x+1}{x-3}$, is analytic at $x=0$

$(x-0)^2Q(x)=-\dfrac{2x}{(x-3)}$, is analytic at $x=0$

$\therefore$ $x=0$ is a regular singular point.

Again, $(x-3)P(x)=\dfrac{x+1}{x}$, is analytic at $x=3$

$(x-3)^2Q(x)=-\dfrac{2(x-3)}{x}$, is analytic at $x=3$

$\therefore$ $x=3$ is a regular singular point.

**3.** $x^2(x+1)^2 y'' + 2(x^2-1) y' - 3y = 0$

Its normalised form is

$$y'' + \frac{2(x^2-1)}{x^2(x+1)^2} y' - \frac{3}{x^2(x+1)^2} y = 0$$

Here,

$$P(x) = \frac{2(x^2-1)}{x^2(x+1)^2}, \text{ is not analytic at } x = 0, -1$$

$$Q(x) = -\frac{3}{x^2(x+1)^2}, \text{ is not analytic at } x = 0, -1$$

$\therefore$ $x = 0, -1$ are the singular points.

Now,

$$(x-0)P(x) = \frac{2(x^2-1)}{x(x+1)^2}, \text{ is not analytic at } x = 0$$

$$(x-0)^2 Q(x) = -\frac{3}{(x+1)^2}, \text{ is analytic at } x = 0$$

$\therefore$ $x = 0$ is an irregular singular point.

Again,

$$(x+1)P(x) = \frac{2(x^2-1)}{x^2(x+1)} = \frac{2(x+1)}{x^2}, \text{ is } \cancel{\text{not}} \text{ analytic at } x = -1$$

$$(x+1)^2 Q(x) = -\frac{3}{x^2}, \text{ is analytic at } x = -1$$

$\therefore$ $x = -1$ is a ~~ir~~regular singular point.

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**4.** $(x^3+x^2)y''+(x^2-2x)y'-y=0$

Its normalised form is

$$y''+\frac{x(x-2)}{x^2(x+1)}y'-\frac{1}{x^2(x+1)}y=0$$

$$\Rightarrow y''+\frac{(x-2)}{x(x+1)}y'-\frac{1}{x^2(x+1)}y=0$$

ie Here, $P(x)=\frac{x-2}{x(x+1)}$, is not analytic at $x=0,-1$

$Q(x)=-\frac{1}{x^2(x+1)}$, is not analytic at $x=0,-1$.

ie $\therefore x=0,-1$ are the singular points.

Now, $(x-0)P(x)=\frac{x-2}{x+1}$, is analytic at $x=0$

$(x-0)^2Q(x)=-\frac{1}{x+1}$, is analytic at $x=0$.

ie $\therefore x=0$ is a regular singular point.

Again, $(x+1)P(x)=\frac{(x-2)}{x}$, is analytic at $x=-1$

$(x+1)^2Q(x)=-\frac{x+1}{x^2}$, is analytic at $x=-1$

$\therefore$ $\therefore x=-1$ is a regular singular point.

5. $(x^4 - 2x^3 + x^2)y'' - 4(x-1)y' - 5x^2 y = 0$

Its normalised form is

$$y'' - \frac{4}{x^2(x-1)}y' - \frac{5}{(x-1)^2}y = 0$$

$x^4 - 2x^3 + x^2 = x^2(x^2 - 2x + 1) = x^2(x-1)^2$

Here, $P(x) = -\frac{4}{x^2(x-1)}$, is not analytic at $x = 0, 1$

$Q(x) = -\frac{5}{(x-1)^2}$, is not analytic at $x = 1$

$\therefore$ $x = 0, 1$ are the singular points.

Now, $(x-0)\cdot P(x) = -\frac{4}{x(x-1)}$, is not analytic at $x = 0$

$(x-0)^2 \cdot Q(x) = -\frac{5x^2}{(x-1)^2}$, is analytic at $x = 0$

$\therefore$ $x = 0$ is an irregular singular point.

Again, $(x-1)\cdot P(x) = -\frac{4}{x^2}$, is analytic at $x = 1$

$(x-1)^2 \cdot Q(x) = -5$, is analytic at everywhere.

$\therefore$ $x = 1$ is a regular singular point.

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6. $(x^3+x^2)y'' + (x^2-2x)y' + 4y = 0$

$\Rightarrow x^2(x+1)y'' + x(x-2)y' + 4y = 0$

Its normalised form is

$$y'' + \frac{(x-2)}{x(x+1)} + \frac{4}{x^2(x+1)}y = 0$$

Here, $P(x) = \dfrac{x-2}{x(x+1)}$, is not analytic at $x=0,-1$

$Q(x) = \dfrac{4}{x^2(x+1)}$, is not analytic at $x=0,-1$

$\therefore x=0,-1$ are the singular points.

Now, $(x-0)\cdot P(x) = \dfrac{x-2}{x+1}$, is analytic at $x=0$

$(x-0)^2\cdot Q(x) = \dfrac{4}{x+1}$, is analytic at $x=0$

$\therefore x=0$ is a regular singular point.

Again, $(x+1)\cdot P(x) = \dfrac{x-2}{x}$, is analytic at $x=-1$

$(x+1)^2\cdot Q(x) = \dfrac{4(x+1)}{x^2}$, is analytic at $x=-1$

$\therefore x=-1$ is a regular singular point.

**7.** $(x^5+x^4-6x^3)y''-x^2y'-2(x-2)y=0$

Its normalised form is

$$y''-\frac{1}{x(x-2)(x+3)}y'-\frac{2}{x^3(x+3)}y=0$$

$x^5+x^4-6x^3$
$=x^3(x^2+x-6)$
$=x^3(x-2)(x+3)$

Here, $P(x)=-\frac{1}{x(x-2)(x+3)}$, is not analytic at $x=0,2,-3$

$Q(x)=-\frac{2}{x^3(x+3)}$, is not analytic at $x=0,-3$

$\therefore$ $x=0,2,-3$ are the singular points.

Now, $(x-0)\cdot P(x)=-\frac{1}{(x-2)(x+3)}$, is analytic at $x=0$

$(x-0)^2\cdot Q(x)=-\frac{2}{x(x+3)}$, is not analytic at $x=0$

$\therefore x=0$ is an irregular singular point.

Again, $(x-2)\cdot P(x)=-\frac{1}{x(x+3)}$, is analytic at $x=2$

$(x-2)^2\cdot Q(x)=\frac{2(x-2)^2}{x^3(x+3)}$, is analytic at $x=2$

$\therefore x=2$ is a regular singular point.

Also, $(x+3)\cdot P(x)=-\frac{1}{x(x-2)}$, is analytic at $x=-3$

$(x+3)^2\cdot Q(x)=-\frac{2(x+3)}{x^3}$, is analytic at $x=-3$

$\therefore x=-3$ is a regular singular point.

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**B** Find the nature of the point $x=0$ for the ODE.

ⓐ $(x+1)xy'' + (\sin x)y = 0$

Its normalised form is

$$y'' + \frac{\sin x}{(x+1)\cdot x}\, y = 0$$

Here, $P(x) = 0$

$$Q(x) = \frac{\sin x}{(x+1)x} = \frac{\frac{x}{\lfloor 1} - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots\cdots}{x(x+1)}$$

$$= \frac{\frac{1}{\lfloor 1} - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} - \cdots\cdots}{x+1}$$

so, clearly at $x=0$, both $P(x)$ and $Q(x)$ are analytic.

$\therefore x=0$ is an ordinary point.

b. $2xy'' - (\cos x)y = 0$

Its normalised form is

$$y'' - \frac{\cos x}{2x}\, y = 0$$

Here, $P(x) = 0$

$$Q(x) = -\frac{\cos x}{2x}$$

$$= -\frac{\left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots\cdots\right)}{2x}$$

so, clearly at $x=0$, $P(x)$ is analytic but $Q(x)$ is not analytic.

$\therefore x=0$ is a singular point.

Now, $(x-0)\cdot P(x) = 0$, is analytic at $x=0$

$(x-0)^2\cdot Q(x) = -\frac{1}{2}x\cos x$, is analytic at $x=0$

$\therefore x=0$ is a regular singular point.

c. $xy'' + 2xy' + (\tan x)y = 0$

Its normalised form is

$$y'' + 2y' + \frac{\tan x}{x} y = 0$$

Here, $P(x) = 2$ , is analytic at $x=0$

$$Q(x) = \frac{\tan x}{x}$$

$$= \frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots\cdots}{x}$$

$$= \frac{1 + \frac{x^2}{3} + \frac{2x^4}{15} + \cdots\cdots}{1},$$

is analytic at $x=0$

$\therefore x=0$ is an ordinary point.

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d. $(x^2-x)y'' - xy' + \log(1+x)y = 0$

Its normalised form is

$$y'' - \frac{1}{x-1}y' + \frac{\log(1+x)}{x(x-1)}y = 0$$

Here, $P(x) = -\frac{1}{x-1}$, is analytic at $x=0$

$$Q(x) = \frac{\log(1+x)}{x(x-1)}$$

$$= \frac{\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\cdots}{x(x-1)}$$

$$= \frac{\frac{1}{1} - \frac{x}{2} + \frac{x^2}{3} - \cdots\cdots}{x-1}$$, is analytic at $x=0$

$\therefore x=0$ is an ordinary point.

e. $x^2y'' + y' + (\sinh x)y = 0$

Its normalised form is

$$y'' + \frac{1}{x^2}y' + \frac{\sinh x}{x^2}y = 0$$

Here, $P(x) = \frac{1}{x^2}$, is not analytic at $x=0$

$$Q(x) = \frac{\sinh x}{x^2}$$

$$= \frac{x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \frac{x^7}{\lfloor 7} + \cdots}{x^2}$$

$$= \frac{1 + \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} + \frac{x^6}{\lfloor 7} + \cdots}{x},$$

is not analytic at $x=0$

$\therefore x=0$ is a singular point.

Now, $(x-0)\cdot P(x) = \frac{1}{x}$, is not analytic at $x=0$.

$(x-0)^2 \cdot Q(x) = \sinh x$, is analytic at $x=0$

$\therefore x=0$ is an irregular singular point.