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UNIT-III

~: Power Series Solution :~

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SERIES SOLUTION OF ODE

err ● **Analytic function**: A function $f(x)$ will be analytic at a point x_0 if $f'(x)$ exist for all $x \in (x_0 - \delta, x_0 + \delta)$, $\delta > 0$

i.e. $f(x)$ is differentiable at a neighbourhood of x_0 ($\because$ Deleted neighbourhood নয় তাহলে x_0 point টেও differentiable হবে)

Examples:

① All polynomial functions are analytic everywhere.

② e^x, $\sin x$, $\cos x$ $\longrightarrow$ are analytic.

③ $f(x) = \dfrac{x-1}{x^2 - 5x + 6} = \dfrac{x-1}{(x-2)(x-3)}$ is analytic except $x = 2$ and $x = 3$ points.

● **Ordinary point**:

consider a second order homogeneous linear differential equation

$$P_0(x) \frac{d^2 y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x) y = 0 \qquad (1)$$

Let, it has no solution, which can be expressed as finite linear combination of elementary functions, but it is possible to have a power series solution.

It is good for the question what is series solution?

The equation (1) can be written as 1.

$$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{——— (2)}$$

where, $P(x) = \dfrac{P_1(x)}{P_0(x)}$

$Q(x) = \dfrac{P_2(x)}{P_0(x)}$ (2)

A point x_0 is said to be an ordinary point of the ODE (2) if $P(x)$ and $Q(x)$ are both analytic at x_0.

Example:

$$\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} + (x-2)y = 0$$

Here, $P(x) = x-1$ and $Q(x) = x-2$ both (3)
are analytic everywhere ($\because$ They are polynomial)
Therefore all the points are ordinary point of the ODE.

(4)

1. $(x-1)(x-2)\frac{d^2y}{dx^2}+\frac{1}{x-1}\frac{dy}{dx}+\frac{1}{x-2}y=0$

$$\Rightarrow \frac{d^2y}{dx^2}+\frac{1}{(x-1)^2(x-2)}\frac{dy}{dx}+\frac{1}{(x-1)(x-2)^2}y=0$$

Remarks: All the points are ordinary point except $x=1$ and $x=2$ point.

② * লক্ষ্য কর- নির্ভর টা কিন্তু শুধুমাত্র $P(x)$ বা শুধুমাত্র $Q(x)$ এর উপর নয়, overall $P(x)$ and $Q(x)$ এর উপর নির্ভর করে।

$$\cancel{(x-1)(x-2)\frac{d^2y}{dx^2}+\frac{1}{x-1}\frac{dy}{dx}+\frac{1}{x-2}y=}$$

$$(x-1)(x-2)\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}+(x-2)y=0$$

$$\Rightarrow \frac{d^2y}{dx^2}+\frac{1}{x-2}\frac{dy}{dx}+\frac{1}{x-1}y=0$$

Remarks: All the points are ordinary point except $x=1$ and $x=2$ point.

③ $\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}+\frac{1}{x-2}y=0$

Remarks: All the points are ordinary point except $x=2$ point.

④ $\frac{1}{(x-1)(x-2)}\frac{d^2y}{dx^2}+\frac{1}{x-1}\frac{dy}{dx}+\frac{1}{x-2}y=0$

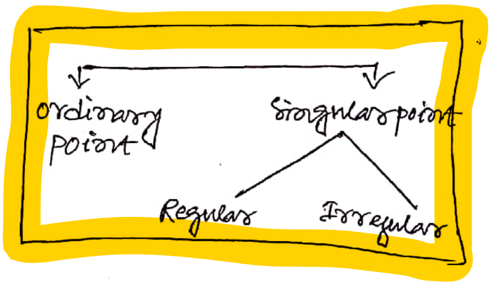
$$\Rightarrow \frac{d^2y}{dx^2}+(x-2)\frac{dy}{dx}+(x-1)y=0$$

Remarks: All the points are ordinary point.

- **Singular point**: A point x_0 is said to be a singular point of the equation ② if either or both of the function $P(x)$ and $Q(x)$ is not analytic at the point x_0

- **Regular singular point**: A singular point x_0 of the equation ② will be regular singular point when $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are both analytic at x_0.

- **Irregular singular point**: If at least one of $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ is not analytic at x_0 then x_0 is called irregular singular point.



Example: cosider the ODE

$$(x+1)\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{x^2(x+1)^2}\,y = 0$$

Its normalised form is

$$\frac{d^2y}{dx^2} + \frac{1}{(x+1)}\frac{dy}{dx} + \frac{1}{x^2(x+1)^3}\,y = 0$$

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$\therefore P(x) = \dfrac{1}{x+1}$, is not analytic at $x=-1$

$\therefore Q(x) = \dfrac{1}{x^2(x+1)^3}$, is not analytic at $x=0,-1$

Now, $(x-0)P(x) = \dfrac{x}{x+1}$

$(x-0)^2 Q(x) = \dfrac{1}{(x+1)^3}$

both are analytic at $x=0$

মনে হতে পারে $x=-1$ এর জন্য তো disturb but এখানে চিন্তা করা হবে only 0 এর জন্য, আর ওর জন্য এখানে disturb নেই

$\therefore x=0$ is a regular singular point.

Again, $(x+1)P(x) = 1$, is analytic at everywhere or $x=-1$.

$(x+1)^2 Q(x) = \dfrac{1}{(x+1)x^2}$, is not analytic at $x=-1$

$\therefore x=-1$ is an irregular singular point.

Power Series Solution near the Ordinary point

Consider the ODE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{—— (1)}$$

Let x_0 be an ordinary point of this equation.

Let us consider the power series

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{—— (2)}$$

Differentiating (2) w.r.t. x twice we get

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n (x-x_0)^{n-2}$$

putting these in (1) and simplifying we get

$$c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \cdots$$
$$\cdots + c_n(x-x_0)^n + \cdots = 0 \quad \left(\text{i.e } 0 + 0(x-x_0) + 0.(x-x_0)^2 + \cdots\right) \quad \text{—— (3)}$$

where $c_0, c_1, c_2 \cdots\cdots$ are functions of $a_0, a_1, \cdots\cdots$

It is an identity

so, $c_0 = 0$, $c_1 = 0$, ……, $c_n = 0$ — solving we get the coefficients $a_0, a_1, \ldots, a_n, \ldots$ and we can get the power series solⁿ from (2) of the equation (1).

(a)
(b)
(c)
(d)
solⁿ

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Problems

Locate and classify the singular point of the equations

(a) $(x^2+x)\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + 2y = 0$

(b) $x^2\frac{d^2y}{dx^2} + (3x-1)\frac{dy}{dx} + y = 0$

(c) $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - xy = 0$

(d) $x^2(x+1)^2\frac{d^2y}{dx^2} + (x^2-1)\frac{dy}{dx} + 2y = 0$

Solⁿ: (a) $(x^2+x)\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + 2y = 0$

Its normalised form is

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2}{x(x+1)}y = 0$$

Here, $P(x) = -\frac{2}{x}$, is not analytic at $x=0$

$Q(x) = \frac{2}{x(x+1)}$, is not analytic at $x=0,-1$

$(x-x_0)^2$

$\therefore x = 0, -1$ are the singular point.

Now, $(x-0)P(x) = -2$, is analytic everywhere or $x=0$

$(x-0)^2Q(x) = \frac{2x}{x+1}$, is analytic at $x=0$

$\therefore x=0$ is a regular singular point.

Also, $(x+1)P(x) = -\frac{2(x+1)}{x}$, is analytic at $x=-1$

$(x+1)^2Q(x) = \frac{2(x+1)}{x}$, is analytic at $x=-1$

$\therefore x=-1$ is also regular singular point.

Sol^n (b) $x^2 \frac{d^2y}{dx^2} + (3x-1)\frac{dy}{dx} + y = 0$

Its normalised form is

$$\frac{d^2y}{dx^2} + \frac{3x-1}{x^2}\frac{dy}{dx} + \frac{1}{x^2}y = 0$$

Here, $P(x) = \frac{3x-1}{x^2}$, is not analytic at $x=0$

$Q(x) = \frac{1}{x^2}$, is not analytic at $x=0$

$\therefore x=0$ is the singular point.

Now, $(x-0)P(x) = \frac{3x-1}{x}$, is not analytic at $x=0$

$(x-0)^2 Q(x) = 1$, is analytic at $x=0$

$\therefore x=0$ is ~~not~~ is regular singular point.

Sol^n (c) $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - xy = 0$

Its normalised form is

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{x}{x^2+1}y = 0$$

Here, $P(x) = \frac{x}{x^2+1}$, is analytic at everywhere.

$Q(x) = -\frac{x}{x^2+1}$, is analytic at everywhere.

Therefore all the points are ordinary point of the ODE.

Sol^n (d) $x^2(x+1)^2\frac{d^2y}{dx^2}+(x^2-1)\frac{dy}{dx}+2y=0$

Its normalised form is

$$\frac{d^2y}{dx^2}+\frac{(x^2-1)}{x^2(x+1)^2}\frac{dy}{dx}+\frac{2}{x^2(x+1)^2}y=0$$

Here, $P(x)=\frac{x^2-1}{x^2(x+1)^2}$, is not analytic at $x=0,-1$

$Q(x)=\frac{2}{x^2(x+1)^2}$, is not analytic at $x=0,-1$

$\therefore$ $x=0,-1$ are the singular point.

Now, $(x-0)P(x)=\frac{x^2-1}{x(x+1)^2}$, is not analytic at $x=0$

$(x-0)^2Q(x)=\frac{2}{(x+1)^2}$, is analytic at $x=0$.

$\therefore$ $x=0$ is irregular singular point.

Again, $(x+1)P(x)=\frac{(x^2-1)}{x^2(x+1)}$, is analytic at $x=-1$

$(x+1)^2Q(x)=\frac{2}{x^2}$, is analytic at $x=-1$

$\therefore$ $x=-1$ is a regular singular point.