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Problems

① Determine the nature of the critical point of the system

$$\frac{dx}{dt} = 2x - 7y, \quad \frac{dy}{dt} = 3x - 8y$$

and determine whether or not the point is stable.

Sol^n: The characteristic equation of the system

$$\begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 6\lambda + 5 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+5) = 0$$

$$\therefore \lambda = -1, \quad \therefore \lambda = -5$$

$\therefore$ The roots are -1 and -5

\# Since the roots are real, unequal and of the same sign

we conclude that the critical point $(0,0)$ of the system is node.

\# Since the roots are real and negative the point is asymptotically stable. (sink)

② Determine the nature of the critical point of the system

$$\frac{dx}{dt} = 2x + 4y, \quad \frac{dy}{dt} = -2x + 6y$$

Sol^n: The characteristic equation of the system

$$\begin{vmatrix} 2-\lambda & 4 \\ -2 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 20 = 0$$

$$\therefore \lambda = \frac{8 \pm \sqrt{64-80}}{2} = 4 \pm 2i$$

$\therefore$ The roots are $4+2i$ and $4-2i$

Since the roots are conjugate complex but not pure imaginary so we conclude that the critical point (0,0) of the system is spiral point.

Stability: Since the real part is positive. So the point is asymptotically unstable.

③ Determine the nature of the critical point of the following linear autonomous systems and also comment on stability of the point.

ⓐ $\frac{dx}{dt} = x+3y$, $\frac{dy}{dt} = 3x+y$

ⓑ $\frac{dx}{dt} = 3x+2y$, $\frac{dy}{dt} = x+2y$

ⓒ $\frac{dx}{dt} = 3x+4y$, $\frac{dy}{dt} = 3x+2y$

ⓓ $\frac{dx}{dt} = 2x+5y$, $\frac{dy}{dt} = x-2y$

ⓔ $\frac{dx}{dt} = 2x-4y$, $\frac{dy}{dt} = 2x-2y$

ⓕ $\frac{dx}{dt} = x-2y$, $\frac{dy}{dt} = 4x+5y$

ⓖ $\frac{dx}{dt} = x-y$, $\frac{dy}{dt} = x+5y$

ⓗ $\frac{dx}{dt} = x+7y$, $\frac{dy}{dt} = 3x+5y$

Solⁿ: ⓐ The characteristic equation of the system

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 - 2\lambda - 8 = 0$

$\Rightarrow (\lambda-4)(\lambda+2) = 0$

$\therefore \lambda = -2, 4$

$\therefore$ The roots are $-2, 4$

Since the roots are real, unequal and of the opposite sign so we conclude that the critical point (0,0) is a saddle point.

The point is unstable.

(b) The characteristic equation of the system

$$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$\Rightarrow (\lambda-1)(\lambda-4) = 0$

$\therefore \lambda = 1, 4$

$\therefore$ The roots are 1 and 4

Since the roots are real, unequal and of the same sign so we conclude that the critical point (0,0) is node.

Since the roots are real and positive so the critical point is asymptotically unstable.

(c) The characteristic equation of the system

$$\begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$

$\Rightarrow (\lambda+1)(\lambda-6) = 0$

$\therefore \lambda = -1, 6$

$\therefore$ The roots are -1 and 6

Since the roots are real, unequal and of the opposite sign so we conclude that the critical point (0,0) is saddle point.

~~Since the roots are~~ The point is unstable.

(d) The characteristic equation of the system

$$\begin{vmatrix} 2-\lambda & 5 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 9 = 0$$

$$\Rightarrow (\lambda+3)(\lambda-3) = 0$$

$$\therefore \lambda = 3, -3$$

$\therefore$ The roots are 3 and -3

Since the roots are real, unequal and of the opposite sign so we conclude that the critical point (0,0) is saddle point.

The point is unstable.

(e) The characteristic equation of the system

$$\begin{vmatrix} 2-\lambda & -4 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 4 = 0$$

$$\therefore \lambda = \pm 2i$$

$\therefore$ The roots are $2i$ and $-2i$

Since the roots are purely imaginary so we conclude that the critical point (0,0) is centre.

The point is stable but not asymptotically.

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(f) The characteristic equation of the system

$$\begin{vmatrix} 1-\lambda & -2 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 13 = 0$$

$$\therefore \lambda = \frac{6 \pm \sqrt{36-52}}{2} = 3 \pm 2i$$

$\therefore$ The roots are $3+2i$ and $3-2i$

Since the roots are conjugate complex but not purely imaginary so we conclude that the critical point $(0,0)$ is spiral point.

Since the real part of the roots are positive so the point is asymptotically unstable.

(g) The characteristic equation of the system

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 6 = 0$$

$$\therefore \lambda = \frac{6 \pm \sqrt{36-24}}{2} = \frac{6 \pm 2\sqrt{3}}{2} = 3 \pm \sqrt{3}$$

(for rough $3+\sqrt{3}$ = positive
$3-\sqrt{3}$ = positive)

$\therefore$ The roots are $3+\sqrt{3}$ and $3-\sqrt{3}$

Since the roots are real, unequal and of the same sign so we conclude that the critical point $(0,0)$ is node.

Since the roots are real and positive so the point is asymptotically unstable.

(h) The characteristic equation of the system

$$\begin{vmatrix} 1-\lambda & 7 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 6\lambda - 16 = 0$$

$$\Rightarrow (\lambda-8)(\lambda+2) = 0$$

$$\therefore \lambda = -2, 8$$

$\therefore$ The roots are -2 and 8

Since the roots are real, unequal and of the opposite sign so we conclude that the critical point $(0,0)$ is saddle point.

The point is unstable.

(4) Consider the linear autonomous system

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = 3x - y$$

i) Find the general solution of this system

ii) Determine the nature of the critical point of the system.

iii) comment on the stability.

Solⁿ (i) The given equations can be written as

$$(D-1)x - y = 0 \quad \text{———— (1)}$$

$$-3x + (D+1)y = 0 \quad \text{———— (2)}, \text{ where } D \equiv \frac{d}{dt}$$

Eliminating y from (1) and (2) we get

$$\cancel{D^2+2D/A\lambda}\ (D^2 - 1 - 3)x = 0$$

$$\Rightarrow (D^2 - 4)x = 0 \quad \text{———— (3)}$$

The auxiliary equation of (3) is

$$m^2 - 4 = 0 \quad \therefore m = \pm 2$$

∴ General solution of (3) is

$$x = c_1 e^{-2t} + c_2 e^{2t}$$, where c_1, c_2 are arbitrary constant. (4)

Now from (1) we get

$$y = (D-1)x$$
$$= (D-1)\{c_1 e^{-2t} + c_2 e^{2t}\}$$
$$= -2c_1 e^{-2t} + 2c_2 e^{2t} - c_1 e^{-2t} - c_2 e^{2t}$$
$$= (-2c_1 - c_1) e^{-2t} + (2c_2 - c_2) e^{2t}$$
$$= -3c_1 e^{-2t} + c_2 e^{2t} \quad (5)$$

(4) and (5) give the general solution of the given system.

(ii) The characteristic equation of the system

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 4 = 0$$
$$\Rightarrow (\lambda+2)(\lambda-2) = 0$$
$$\therefore \lambda = 2, -2$$

∴ The roots are 2 and −2

Since the roots are real, unequal and of the opposite sign so we conclude that the critical point (0,0) is saddle point.

The point is unstable.

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