Determination of critical points

Consider the linear system of ODE

$$\left.\begin{aligned} \frac{dx}{dt} &= ax+by \\ \frac{dy}{dt} &= cx+dy \end{aligned}\right\} \text{———— (1)}$$

where a, b, c, d are real constants.

The origin $(0,0)$ is a critical point or equilibrium point of the system.

We assume that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Note: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ হলে $(0,0)$ is only critical point, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$ হলে, $(0,0)$ ছাড়াও অসংখ্য critical point থাকবে but $(0,0)$ is always a critical point.

and hence $(0,0)$ is the only critical point of the system. We found the solution of the given equation is of the form

$$\left.\begin{aligned} x &= Ae^{\lambda t} \\ y &= Be^{\lambda t} \end{aligned}\right\} \text{———— (2)}$$

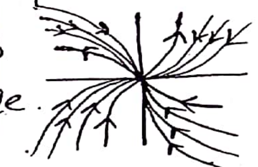
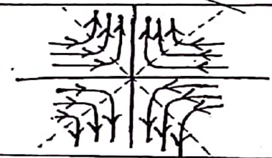
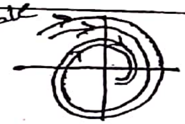
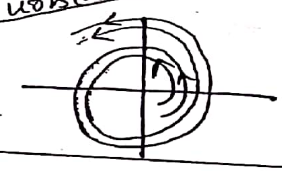
where λ must satisfy the quadratic equation $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$ ———— (3)

This equation is called the characteristic equation of the system. The nature of critical points of (1) depends on the roots of the equations (3).

Let the roots of (3) are λ_1 and λ_2

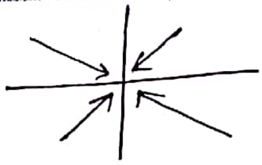
Note: $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$

———— comes from $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

Nature of λ_1 and λ_2	Nature of critical point (0,0)	Stability of critical point.	picture
(i) Real, unequal, same sign.	(i) Node	Asymptotically stable if the roots are '–ve', asymptotically unstable if the roots are '+ve'.	
(ii) Real, unequal, opposite sign.	(ii) Saddle point	unstable	
(iii) Real and equal.	(iii) Node	Asymptotically stable if the roots are '–ve' and asymptotically unstable if the roots are '+ve'.	I know.
(iv) conjugate complex but not pure imaginary. like this $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$	(iv) Spiral point	Asymptotically stable if real part is negative ('–ve') and unstable if real part is positive ('+ve').	For stable  For unstable 
(v) pure imaginary (only imaginary part)	(v) centre	Stable but not asymptotically. So, it does not called sink.	I know.

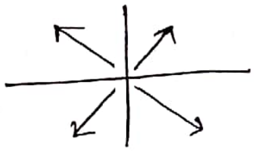
(A Gangopadhyay)

- Stable → (Sink)



(Sink)

- Source → (Unstable)



(source)

- What is asymptotically stable - unstable?

suppose it is a curve.

when both λ is '$-$ve' then, $x = Ae^{-\lambda_1 t}$

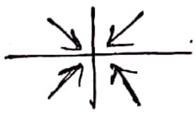
$y = Be^{-\lambda_2 t}$

as $t \to \infty$ both zero.

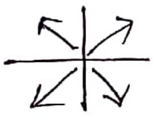
so, when both λ is negative then as $t \to \infty$ the curve ~~[illegible]~~ touch

but when both λ is positive then as $t \to \infty$ the curve does not touch. (both not zero)

→ In this situation stable and it is known as sink



→ In this situation unstable and it is known as source



- Saddle point — ~~[illegible]~~ always unstable.

⇒ In this case one λ is '$-$ve' and one λ is '$+$ve' (opposite sign)

Let, $\lambda_1 =$ '$-$ve', $\lambda_2 =$ '$+$ve'

then $x = Ae^{-\lambda_1 t}$, $y = Be^{\lambda_2 t}$

so, as $t \to \infty$ one tends to zero and other tends to ∞.

so, ultimately unstable.

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