UNIT-III

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~ : Phase-Plane : ~

S.L. ROSS (for better discussion) [Follow Bej-Mukherjee for Definition of non-linear D.E] [redacted]

Equilibrium points and phase plane (part of non-linear D.E)

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + 5y = 6$$

Derivative এর সঙ্গে dependent variable গুণ থাকলে সমীকরণ — this is called non-linear Differential equation.

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The second order non-linear ODE is of the form —

$$\boxed{\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}\right)} \quad \text{—— (1)}$$

For example:

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0 \quad \text{(Vander-Pol equation)}$$

where μ is positive constant.

We shall consider this equation at a larger stage of the value of t.

Let us suppose equation (1) describes a certain dynamical system. The state of this system at time t is determined by the value of x (position) - (physical meaning) and $\frac{dx}{dt}$ (velocity) - physical meaning.

The plane of the variables x and $\frac{dx}{dt}$ is called a phase plane.

Let us assume $y = \frac{dx}{dt}$, then equation (1) reduces to

$$\left.\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= F(x, y) \end{aligned}\right\} \quad \text{—— (2)}$$

Here t is a parameter such that the curves will appear in xy-plane.

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Generally we shall consider system of equation of the form

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y) \end{aligned}\right\} \quad \text{--- (3)}$$

where P and Q have continuous partial derivative for x, y. This system is known as autonomous system.

't' missing in the R.H.S

$$\begin{cases} \frac{dx}{dt} = P(x,y) \\ \frac{dy}{dt} = Q(x,y) \end{cases}$$

Equilibrium point

Definition: Given the autonomous system

$$\frac{dx}{dt} = P(x,y)$$

$$\frac{dy}{dt} = Q(x,y)$$

a point (x_0, y_0) at which both $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$, is called a critical point or equilibrium point or singular point of the equation.

$\boxed{\text{Note}}$ From the above equation eliminating t we get

$$\boxed{\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}}$$

at a point (x_0, y_0) both P and Q are zero i.e the slope of the tangent to the path is indeterminate. Such a ~~p~~ point is known as critical point or equilibrium point or singular point.

For example:

consider the linear autonomous system:-

$$\left.\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x\end{aligned}\right\} \quad \text{——— (a)}$$

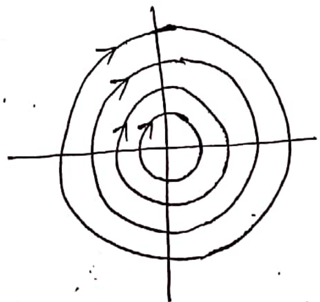
Eliminating 't' we get

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}} \quad \text{——— (b)}$$

solving, we get

$x^2+y^2 = c^2$ — which give one parameter family of curve. ($c \rightarrow$ parameter)

Equation (b) gives the one parameter family of paths in the xy - phase plane.



Definition: A critical point (x_0, y_0) is called isolated if (x_0, y_0) is the only critical point within the circle

$$(x-x_0)^2+(y-y_0)^2 = r^2$$

so, in the above example the point $(0,0)$ is the isolated critical point.

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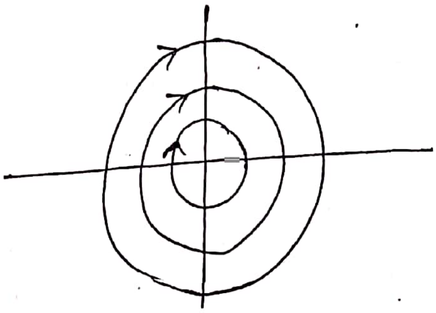
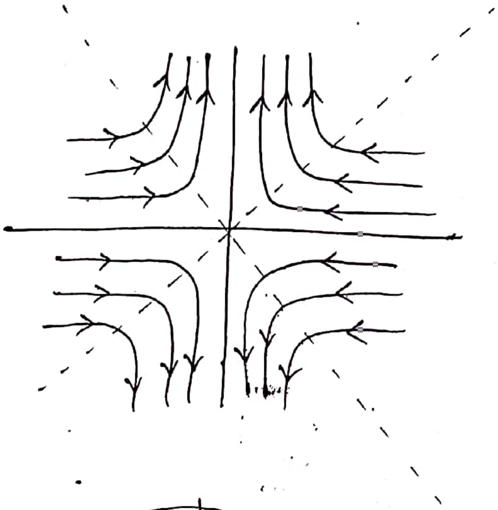
Types of critical point

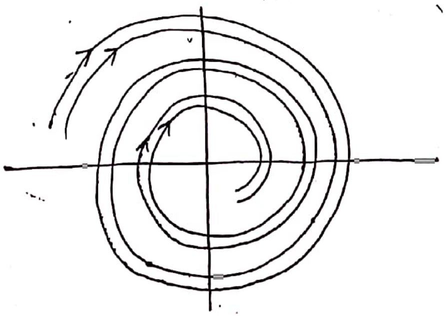
(i) Centre:



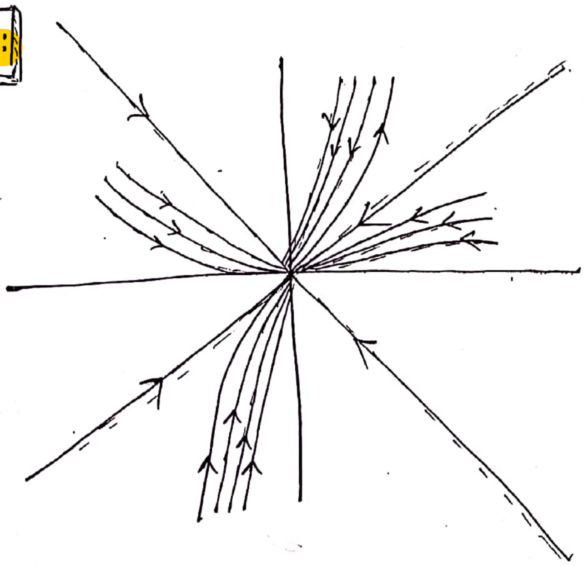
(ii) Saddle point:



(iii) Spiral point:



(iv) Node:



Q/P

$P \to 0$ as $t \to \pm\infty$

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