class – 3 (ipsita.rajwar@gmail.com)

Homogenean Linear systems of differential equations with constant coefficients :

We will concentrate most of our attention on systems of homogeneous linear equations with ~~coe~~ constant coefficients — that is, system of the form

$$X' = AX \quad \ldots\ldots (1)$$

Where A is constant $n \times n$ matrix.

- If $n=1$, the system reduces to a single first order eqn

$$\frac{dx}{dt} = ax$$

whose soln is $x = ce^{at}$.

- The case $n=2$ is particularly important and we will focus on it.

The homogeneous linear system of a 1st order (ODE) differential equation in two variables x & y has the normal form

$$\left.\begin{aligned} \frac{dx}{dt} &= a_{11}x + a_{12}y \\ \frac{dy}{dt} &= a_{21}x + a_{22}y \end{aligned}\right\}$$

here a_{ij}'s are constant.

There are several ways to find the solutions of this system, and we will discuss ~~this~~ these ways through proper examples.

Consider
$$\left.\begin{aligned} \frac{dx}{dt} &= x + 3y \\ \frac{dy}{dt} &= x - y \end{aligned}\right\} \text{———} \circledast.$$

(1) First way → Operator method

$$\frac{dx}{dt} = x + 3y$$
$$\frac{dy}{dt} = x - y$$

using notation $\frac{d}{dt} \equiv D$ we have

$$\left.\begin{aligned} Dx &= x + 3y \\ Dy &= x - y \end{aligned}\right\}$$

or,
$$(D-1)x - 3y = 0 \quad \text{--- (i)}$$
$$-x + (D+1)y = 0 \quad \text{--- (ii)}$$

apply $(D+1)$ on (i) & multiply by 3 with eqn (ii).

$$(D+1)(D-1)x - 3(D+1)y = 0$$
$$-3x + 3(D+1)y = 0$$

adding $[(D+1)(D-1) - 3x]x = 0.$

$$(D^2 - 4)x = 0$$

A.E $\quad m^2 - 4 = 0 \Rightarrow m = \pm 2.$

C.F $\quad x = c_1 e^{2t} + c_2 e^{-2t}$

$\therefore$ general soln is $x = c_1 e^{2t} + c_2 e^{-2t}$

$$\therefore \frac{dx}{dt} = 2c_1 e^{2t} - 2c_2 e^{-2t}$$

from (i) we have $\quad 3y = \frac{dx}{dt} - x$

$$\therefore 3y = 2c_1 e^{2t} - 2c_2 e^{-2t} - c_1 e^{2t} - c_2 e^{-2t}$$

$$\therefore y = \frac{1}{3}\left(c_1 e^{2t} - 3c_2 e^{-2t}\right) = \frac{1}{3}c_1 e^{2t} - c_2 e^{-2t}$$

Therefore the whole solution to the system is then

$$\left.\begin{aligned} x &= c_1e^{2t}+c_2e^{-2t} \\ y &= \tfrac{1}{3}c_1e^{2t}-c_2e^{-2t} \end{aligned}\right\}.$$

(2) Second way $\rightarrow$ **Eliminating variables**.

$$\frac{dx}{dt}=x+3y \quad \text{——(i)}$$

$$\frac{dy}{dt}=x-y \quad \text{——(ii)}$$

differentiating (i) w.r.t 't', we get

$$\frac{d^2x}{dt^2}=\frac{dx}{dt}+3\frac{dy}{dt}$$

$$\Rightarrow \frac{d^2x}{dt^2}=\frac{dx}{dt}+\cancel{8}\left(4x-\frac{dx}{dt}\right)$$

(from (v))

$$\Rightarrow \frac{d^2x}{dt^2}=4x$$

$$\Rightarrow \frac{d^2x}{dt^2}-4x=0.$$

from (i),

$$3y=\frac{dx}{dt}-x \quad \text{——(iii)}$$

from (ii)

$$y=x-\frac{dy}{dt} \quad \text{——(iv)}$$

$$\therefore \frac{dx}{dt}-x=3\left(x-\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{dx}{dt}-x=3x-3\frac{dy}{dt}$$

$$\Rightarrow 3\frac{dy}{dt}=4x-\frac{dx}{dt} \quad \text{——(v)}$$

A.E $\quad m^2-4=0 \Rightarrow m=\pm 2$

C.F $\quad x=c_1e^{2t}+c_2e^{-2t}$

general solⁿ $\quad x=c_1e^{2t}+c_2e^{-2t}$

$$\frac{dx}{dt}=2c_1e^{2t}-2c_2e^{-2t}$$

from (iii) $y = \frac{1}{3}\left[\frac{dx}{dt} - x\right]$

$$= \frac{1}{3}\left[2c_1e^{2t} - 2c_2e^{-2t} - c_1e^{2t} - c_2e^{-2t}\right]$$

$$= \frac{1}{3}\left[c_1e^{2t} - 3c_2e^{-2t}\right]$$

$\therefore$ The solution of the system is

$$\left.\begin{aligned} x &= c_1e^{2t} + c_2e^{-2t} \\ y &= \frac{1}{3}c_1e^{2t} - c_2e^{-2t} \end{aligned}\right\}$$

Systems of Linear Algebraic Equations; Linear independence, Eigen values, Eigen vectors :

In this section we summarize some result from Linear algebra that are important for the solution of systems of Linear differential equations.

Systems of linear algebraic equation : A set of n simultaneous linear algebraic equations in n variables

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}\right\} \quad (1)$$

can be written as

$$AX = b, \quad \cdots\cdots \quad (2)$$

where A is $n \times n$ matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $b = 0$, the system is said to be homogeneous, otherwise it is non homogeneous.

If A is nonsingular – i.e, if $\det A \neq 0$, then there is a unique solⁿ of the system (2).

if $\det A = 0$, the solⁿ of (2) either do not exists, or do exists but not unique.

if $\det A = 0$, for the homogeneous system $AX = 0$, then it has infinity many solution.

Linear independence : A set of k vectors $\{x_1, \cdots, x_k\}$ is said to be linearly dependent if there exists a set of numbers $c_1, \cdots, c_k$, at least one of which is non zero, such that

$$c_1 x_1 + \cdots + c_k x_k = 0 \quad \cdots\cdots (i)$$

On the other hand, if the only set $c_1, \cdots, c_k$ for which eqn (i) is satisfied is $c_1 = c_2 = \cdots c_k = 0$, then $\{x_1, \cdots, x_k\}$ is said to ~~be~~ linearly independent set of vectors.

Eigenvalue & eigenvector

The equation $Ax = y$ can be viewed as a linear transformation that maps a given vector x into a new vector y.
now if we want $y = \lambda x$ where λ is a scalar, and seek soln of the equation

$$Ax = \lambda x$$

or

$$(A - \lambda I)x = 0 \quad \cdots \ (ii)$$

This eqn (ii) has nonzero soln if and only if λ is chosen so that

$$\det(A - \lambda I) = 0 \quad \cdots \ (iii)$$

The value for which eqn (iii) is satisfy is called Eigen value of the matrix A, and the non zero solution of (ii) that are obtained by using such a value of λ are called the Eigenvectors corresponding to that eigenvalue.

※ The Equation (iii), i.e $\det(A - \lambda I) = 0$ is called the characteristic equation of the matrix A.