Basic Theory of linear systems in normal form:

The Normal form of a system of n linear differential equation can be expressed as ⊛

$$X'(t) = A(t)\,X(t) + F(t),$$

if $F(t)=0$, then we call it homogeneous system, otherwise the system is nonhomogeneous.

The initial value problem for the normal system ⊛ is the problem finding a differentiable vector function $X(t)$ that satisfies the system on an interval I and also satisfies the initial condition $X(t_0) = X_0$, where t_0 is a given point on I and $X_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ \vdots \\ x_{n,0} \end{bmatrix}$ is a given vector.

Now we have some basic results as follows:

- **Theorem** (Existence and Uniqueness)

  If the entries of the matrices $A(t)$ and $F(t)$ be continuous on an open interval I that contains the point t_0, then for any choice of the initial vector X_0, there exist a unique solution $X(t)$ on the whole interval I to the initial value problem

$$X'(t) = A(t)\,X(t) + F(t), \quad X(t_0) = X_0$$

- **Definition:**

  1) Linearly independent & Linearly dependent:

  The vector functions $X_1(t), X_2(t), \cdots, X_n(t)$ are linearly dependent on an interval I if there exists numbers $c_1, c_2, \cdots c_n$, not all zero, such that

$$c_1 X_1(t) + c_2 X_2(t) + \cdots + c_n X_n(t) = 0, \quad \forall\, t \in I.$$

  If this indentity is satisfied only if

$$c_1 = c_2 = \cdots = c_n = 0,$$ the vector functions $X_1(t), X_2(t), \cdots X_n(t)$ are called linearly independent on I.

- Any system of n linearly independent solutions $x_1(t), x_2(t), \dots x_n(t)$ is called fundamental system of solutions.

2) **Wronskian**:

The Wronskian of n vector functions $x_1(t), x_2(t), \dots x_n(t)$ is defined to be the real-valued function

$$W(t) = W[x_1, x_2, \dots, x_n](t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \vdots & x_{2n}(t) \\ \vdots & \cdots & \cdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \vdots & x_{nn}(t) \end{vmatrix}$$

Where $x_{ij}(t)$ are the coordinates of the vector function $x_1(t), \dots x_n(t)$.

$\boxed{0}$ The set of solution vectors $x_1(t), \dots x_n(t)$ is linearly independent on I if & only if their Wronskian $W(t)$ is non zero for any t in the interval.

- **Theorem**: (Representation of solutions)

Let X_p be a particular solution to the nonhomogeneous system $X'(t) = A(t)X(t) + F(t)$ on the interval I, & let $\{x_1, x_2, \dots, x_n\}$ be a fundamental solution set on I for the corresponding homogeneous system

$$x'(t) = A(t)x(t)$$

Then every solution to the nonhomogeneous system on I can be expressed in the form

$$x(t) = x_p + c_1 x_1(t) + c_2 x_2(t) + \dots + c_n x_n(t)$$

Where c_i's are constants.

Ex.

1) ~~Ex~~ Let the vector functions given by $x_1(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}$, $x_2(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$ are solutions to the system $X'(t) = A(t)X(t)$, check whether they form a fundamental solution set. If yes, then find the general solution.

Ans:

The Wronskian for the vector function $X_1(t)$ & $X_2(t)$ is

$$W(X_1, X_2) = \begin{vmatrix} 2e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{vmatrix} = -2e^{2t} - e^{2t}$$
$$= -3e^{2t} \neq 0 \ \forall t$$

hence the solⁿs $\{X_1(t), X_2(t)\}$ are linearly independent which means they form a fundamental solⁿ set.

⊙ The general solⁿ for $X'(t) = A(t)X(t)$ is given by

$$X(t) = c_1 X_1(t) + c_2 X_2(t)$$
$$= c_1 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Ex.

(2) Consider the linear system

$$X' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} X + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Verify that the vector functions

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t, \quad X_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$ are solⁿ to the homogeneous system on $(-\infty, \infty)$, and that

$$X_p = \frac{3}{2}\begin{bmatrix} te^t \\ te^t \end{bmatrix} - \frac{1}{4}\begin{bmatrix} e^t \\ 3e^t \end{bmatrix} + \begin{bmatrix} t \\ 2t \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ is a

Particular solⁿ of it. Also find the general solⁿ.

→ Home Work!

Exercise:

(3) Show that $x = t+1$, $y = -5t-2$ is a particular solⁿ of

$$\frac{dx}{dt} = 5x + 2y + 5t$$
$$\frac{dy}{dt} = 3x + 4y + 17t$$

and give the general solⁿ.