ipsita.rajwar@gmail.com **Linear Differential Equation**

Unit-II

Class-1

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⊙ What to study & why?

In this chapter, we will discuss about systems of linear differential equations which is basically a set of linear equations relating a group of functions to their derivatives.

There are many physical real life problems that involving find several unknown functions simultaneously. For example, electrical network have this character. (For more details consult books!)

[Elementary differential equations & Boundary value problems — E. Boyce, C. Diprima]

• Linear differential Equation:

A linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives.

So it is an equation of the form

$$a_n(x)y^{n} + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y' + a_0(x)y + b(x) = 0$$

where $a_i(x)$ $(i=1,2,\ldots n)$ & $b(x)$ are differentible functions that do not need to be linear. $y^{(n)}$ are the derivatives of unknown function y w.r.t variable x.

— This is an ordinary differential equation (ODE).

We may characterised linear differential equation as follows —

Linear Differential Equation

- ODE
  - Homogeneous
  - Non homogeneous
- PDE
  - Homogeneous
  - non homogeneous

1. A Linear system of 1st order ordinary differential equations in two variables x & y of the form

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1(t)\,x + a_2(t)\,y + F_1(t)\\ \frac{dy}{dt} &= b_1(t)\,x + b_2(t)\,y + F_2(t)\end{aligned}\right\} \circledast$$

is called the normal form in two variables.

It can also be written in matrix form as

$$X'(t) = A(t)\,X(t) + b(t)$$

where

$$X' = \begin{pmatrix} x \\ y \end{pmatrix}'$$

(' denotes derivative w.r.t t)

$$A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ b_1(t) & b_2(t) \end{pmatrix}$$

$$b(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix}$$

If $F_1(t) = F_2(t) = 0$; then system $\circledast$ is called Homogeneous.

Any solution of the above system will be ordered pair

$$x = f(t)$$
$$y = g(t)$$

on $a \le t \le b$ such that both f & g satisfies the system ⊛ simultaneously.

The Normal form in general for a system of n differential eqn's in n unknowns is given by —

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + F_1$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + F_2$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + F_n$$

Let

$$x_1 = x \Rightarrow \frac{dx_1}{dt} = \frac{dx}{dt}$$

$$x_2 = \frac{dx}{dt} \Rightarrow \frac{dx_2}{dt} = \frac{d^2x}{dt^2}$$

$$x_3 = \frac{d^2x}{dt^2} \Rightarrow \frac{dx_3}{dt} = \frac{d^3x}{dt^3}$$

$$\vdots$$

$$x_n = \frac{d^{n-1}x}{dt^{n-1}} \Rightarrow \frac{dx_n}{dt} = \frac{d^nx}{dt^n}$$

$$\therefore \frac{d^nx}{dt^n} = a_{n1}x + a_{n2}\frac{dx}{dt} + \cdots + a_{nn}\frac{d^{n-1}x}{dt^{n-1}} + F_n$$

Thus, a system of n eqn's can be transformed into a n^{th} order linear differential eqn.

conversely, any general n^{th} order'd linear differential eqn given by

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1 x = F(t)$$

can be transformed into a system of n diff'al equation by the above transformation.

$$x = x_1$$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_3$$

$$\vdots$$

$$\frac{dx_{n-1}}{dt} = x_n$$

$$\frac{dx_n}{dt} = -a_1 x_1 - a_2 x_2 \cdots - a_n x_n + F(t)$$

Method of solution:

Let $\frac{d}{dt} \equiv D$

$\therefore$ we have

$$F_1(D)x + F_2(D)y = T_1 \longrightarrow (1)$$

$$\phi_1(D)x + \phi_2(D)y = T_2 \longrightarrow (2)$$

F_1, F_2, ϕ_1, ϕ_2 being all rational functions in D with constant co-efficients, T_1, T_2 fn's of the independent variable t.

To solve this, say, ~~we~~ we eliminate y between (1) & (2) to get

$$F(D)x = T(t) \longrightarrow (3)$$

The solⁿ being $x = f(t)$ with no. of arbitrary constants.

Another method:

In this, we ~~we~~ differentiate both (1) & (2) with respect to ~~to~~ 't' to obtain equation in $x, y, \frac{dy}{dt}, \frac{dx}{dt}, \frac{d^2y}{dt^2}, \frac{d^2x}{dt^2}$

From these 4 equⁿs we eliminate say $y, \frac{dy}{dt}, \frac{d^2y}{dt^2}$ to get a 2nd order differential equation in x, we can be solved to get a solⁿ for x.

y is obtained by substituting the value of x obtained.

Example:

Consider the linear differential operator

$$5D^3 + 2D^2 - 3D + 1.$$

If x is a 3 times differentiable funⁿ of t

then

$(5D^3 + 2D^2 - 3D + 1)x$ denotes

$$5\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + x$$

Some properties of Linear differential operator with constant coefficients

Let $L(D) = D^n + a_1 D^{n-1} + \cdots + a_n$ (ai constant)

- **Linearity** u_1 & u_2 are fu^n's & c_i's constant.

$$P(D)(c_1u_1 + c_2u_2) = c_1 P(D)u_1 + c_2 P(D)u_2$$

- **Sum Rule** If $P(D)$ & $q(D)$ are polynomial operator then for any (sufficiently diff'ble) fu^n u,

$$[P(D) + q(D)]u = P(D)u + q(D)u$$

- **Multiplicative Rule:**

If $P(D) = g(D)h(D)$ then

$$P(D)u = g(D)(h(D)u)$$

Example:

Solve the system

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$$

$$2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2$$

using operator method.

Sol^n:

$$2Dx - 2Dy - 3x = t \quad \text{—(i)} \qquad \left[\frac{d}{dt} \equiv D\right.$$

$$2Dx + 2Dy + 3x + 8y = 2 \quad \text{—(ii)}$$

$$\therefore (2D-3)x - 2Dy = t \quad \text{—(iii)}$$

$$(2D+3)x + (2D+8)y = 2 \quad \text{—(iv)}$$

To eliminate y from both the eq^n, we ~~multip~~ apply the operator $(2D+8)$ in (iii) & apply the operator $2D$ in (iv) hence,

$$(2D+8)(2D-3)x - (2D+8)2D\,y = (2D+8)t$$

$$2D(2D+3)x + 2D(2D+8)y = (2D)2$$

adding

$$[(2D+8)(2D-3) + 2D(2D+3)]x = (2D+8)t + (2D)2$$

$$\Rightarrow (8D^2+16D-24)x = 2+8t+0$$

$$\Rightarrow (D^2+2D-3)x = t+\frac{1}{4}$$

$$A.E = m^2+2m-3 = 0$$

$$m = 1, -3$$

$$\therefore C.F = c_1e^{t} + c_2e^{-3t}$$

$$P.I = \frac{1}{(D^2+2D-3)}\left(t+\frac{1}{4}\right)$$

$$= \frac{1}{-3\left(1-\left(\frac{D^2+2D}{3}\right)\right)}\left(t+\frac{1}{4}\right)$$

$$= -\frac{1}{3}\left(1-\left(\frac{D^2+2D}{3}\right)\right)^{-1}\left(t+\frac{1}{4}\right)$$

$$= -\frac{1}{3}\left(1+\frac{D^2+2D}{3}+\cdots\right)\left(t+\frac{1}{4}\right)$$

$$= -\frac{1}{3}\left(t+\frac{2}{3}+\frac{1}{4}\right)$$

$$= -\frac{1}{3}t - \frac{11}{36}$$

$\therefore$ The general solⁿ is

$$x = c_1e^{t} + c_2e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

$$\therefore \frac{dx}{dt} = c_1e^{t} - 3c_2e^{-3t} - \frac{1}{3}$$

now from (1) we have

$$2\frac{dy}{dt} = 2\frac{dx}{dt} - 3x - t$$

$$\Rightarrow 2\frac{dy}{dt} = 2c_1e^{t} - 6c_2e^{-3t} - \frac{2}{3} - 3c_1e^{t} - 3c_2e^{-3t} + \cancel{t} + \frac{11}{12} - \cancel{t}$$

$$\Rightarrow 2\frac{dy}{dt} = -c_1e^{t} - 9c_2e^{-3t} + \frac{3}{12}$$

from (2), we can have

$$y = -\frac{1}{2}c_1e^{t} + \frac{3}{2}c_2e^{-3t} + \frac{1}{8}t + \frac{5}{12}$$

$$\therefore \left.\begin{aligned} x &= c_1e^{t} + c_2e^{-3t} - \frac{1}{3}t - \frac{11}{36} \\ y &= -\frac{1}{2}c_1e^{t} + \frac{3}{2}c_2e^{-3t} + \frac{1}{8}t + \frac{5}{12} \end{aligned}\right\} \text{ the general sol}^n.$$

NOTE:

The no. of arbitrary coefficients for general sol^n depends on the determinant of the operator.

$$\begin{vmatrix} 2D-3 & -2D \\ 2D+3 & 2D+8 \end{vmatrix} = 8D^2 + 16D - 24$$

since this is of order 2, the no of independent constants in general sol^n of the system must also be 2.

Exercises :

1. Transform the given equation into a system of first order equations.

   i) $t^2u'' + tu' + (t^2 - \frac{1}{4})u = 0$

   ii) $u'' + \frac{1}{2}u' + 2u = 0$

   iii) $u'' + \frac{1}{4}u' + 4u = 2\cos 3t$, $u(0)=1, u'(0)=-2$

2. Transform the given system into a single equation of higher order.

   i)
   $$x_1' = 3x_1 - 2x_2$$
   $$x_2' = 2x_1 - 2x_2$$

   ii)
   $$x_1' = -2x_1 + x_2$$
   $$x_2' = x_1 - 2x_2$$

3. Use operator method to solve.

   i)
   $$\frac{dx}{dt} + \frac{dy}{dt} - x - 3y = e^t$$
   $$\frac{dx}{dt} + \frac{dy}{dt} + x = e^{3t}$$

   ii)
   $$\frac{dx}{dt} - 7x + y = 0$$
   $$\frac{dy}{dt} - 2x - 5y = 0$$

   iii)
   $$\frac{dx}{dt} + \frac{dy}{dt} - x = -2t$$
   $$\frac{dx}{dt} + \frac{dy}{dt} - 3x - y = t^2$$