Triple Product

Study material by Dr. P. Mondal

Scalar Triple Product. Let $a = a_1 i + a_2 j + a_3 k$, $b = b_1 i + b_2 j + b_3 k$ and $c = c_1 i + c_2 j + c_3 k$, where i, j, k are unit vectors along the three mutually $\perp$ axes, then the scalar triple product is defined by

$$[a\,b\,c] = a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note: (i) $[abc] = [bca] = [cab]$

(ii) $[abc] = -[bac]$, (iii) $a \cdot (b \times c) = (a \times b) \cdot c$

(iv) The scalar triple product gives the volume of the parallelopiped whose sides are represented by the vectors a, b, c.

Vector triple Product.

Let a, b, c be three vectors. Then the vector triple product is defined by $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$

Note: $(a \times b) \times c = -c \times (a \times b) = (a \cdot c) b - (b \cdot c) a$.

Examples: 1. $[a+b, b+c, c+a] = 2[abc]$

$LHS = (a+b) \cdot \{(b+c) \times (c+a)\} = (a+b) \cdot (b \times c + b \times a + c \times a)$

$= a \cdot b \times c + 0 + 0 + 0 + 0 + b \cdot c \times a = [abc] + [bca] = 2[abc]$

2. $[a \times b, b \times c, c \times a] = [abc]^2$

Let $b \times c = \alpha$. Then $(b \times c) \times (c \times a) = \alpha \times (c \times a) = (\alpha \cdot a) c - (\alpha \cdot c) a$

$= [bca]\, c = [abc]\, c$

Now $LHS = (a \times b) \cdot [abc] c = [abc]\,[abc] = [abc]^2$.

Exercises: 1. Prove $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$

2. Show that $\alpha \times (\beta \times \gamma) + \beta \times (\gamma \times \alpha) + \gamma \times (\alpha \times \beta) = 0$.

3. Show that the four points whose position vectors are $\alpha, \beta, \gamma, \delta$ are coplanar if $[\alpha\beta\gamma] = [\beta\gamma\delta] + [\gamma\alpha\delta] + [\alpha\beta\delta]$

4. If the four vectors a, b, c, d be such that $a + b + c + d = 0$ then show that $\dfrac{|a|}{[\beta\gamma\delta]} = \dfrac{-|b|}{[\gamma\delta\alpha]} = \dfrac{|c|}{[\delta\alpha\beta]} = \dfrac{-|d|}{[\alpha\beta\gamma]}$

where $\alpha, \beta, \gamma, \delta$ are unit vectors along a, b, c, d respectively.

5. Establish the identities:

(i) $2a = i \times (a \times i) + j \times (a \times j) + k \times (a \times k)$

(ii) $[p\,q\,r][a\,b\,c] = \begin{vmatrix} p.a & p.b & p.c \\ q.a & q.b & q.c \\ r.a & r.b & r.c \end{vmatrix}$.

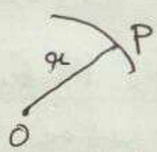
6. Prove that $(\alpha \times \beta)\cdot(\gamma \times \delta) + (\alpha \times \gamma)\cdot(\delta \times \beta) + (\alpha \times \delta)\cdot(\beta \times \gamma) = 0$

Using it to show that $\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B$.

7. Show that $|\alpha \times \beta|^2 |\alpha \times \gamma|^2 - \{(\alpha \times \beta)\cdot(\alpha \times \gamma)\}^2 = |\alpha|^2 [\alpha\beta\gamma]^2$

Differentiation of vectors

Vector function, limit and continuity: Let P be a variable point on a curve in space whose position vector relative to a fixed origin O be r.



If there exists an independent scalar variable t such that for each value of t in a definite domain, we get a definite position of P, i.e. a unique vector r, then r is called a single valued function of t and is represented as $r = f(t)$.

Let P be the point (x, y, z) then we write

$r = x(t)i + y(t)j + z(t)k$, and the specification of the vector function r defines $x, y,$ and z as functions of t.

A vector function $f(t)$ of the scalar ~~func~~ parameter t is said to tend to a limit l as t tends to t_0 if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(t) - l| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

This is expressed as $\lim_{t \to t_0} f(t) = l$.

A vector function $f(t)$ is said to be continuous at $t = t_0$ if $\lim_{t \to t_0} f(t) = f(t_0)$.

Derivative: The derivative of a vector function $r(t)$ of a single parameter t is

$$r'(t) = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} \quad \text{if exists.}$$

If $r(t) = x(t)i + y(t)j + z(t)k$ then $\frac{dr}{dt} = x'(t)i + y'(t)j + z'(t)k$

Proposition. Suppose A, B, C are differentiable vector functions of a scalar t and ϕ is a diff. scalar function of u. Then

(i) $\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt}$

(ii) $\frac{d}{dt}(A\cdot B) = A\cdot\frac{dB}{dt} + \frac{dA}{dt}\cdot B$

(iii) $\frac{d}{dt}(A\times B) = A\times\frac{dB}{dt} + \frac{dA}{dt}\times B$

(iv) $\frac{d}{dt}(\phi A) = \phi\frac{dA}{dt} + \frac{d\phi}{dt}A$

(v) $\frac{d}{dt}[ABC] = [AB\frac{dC}{dt}] + [A\,\frac{dB}{dt}\,C] + [\frac{dA}{dt}\,BC]$

(vi) $\frac{d}{dt}[A\times(B\times C)] = A\times(B\times\frac{dC}{dt}) + A\times(\frac{dB}{dt}\times C) + \frac{dA}{dt}\times(B\times C)$.

Partial derivative of vectors: Suppose A is a vector function of x, y and z, i.e. $A = A(x,y,z)$. The partial derivative of A w.r.t. x is denoted and defined as

$$\frac{\partial A}{\partial x} = \lim_{\Delta x\to 0}\frac{A(x+\Delta x, y, z) - A(x,y,z)}{\Delta x} \text{ if exists.}$$

The other partial derivatives $\frac{\partial A}{\partial y}$ and $\frac{\partial A}{\partial z}$ are similarly defined. Higher order derivatives can be defined as in calculus.

If A has continuous partial derivatives of the second order at least, we have $\frac{\partial^2 A}{\partial x\partial y} = \frac{\partial^2 A}{\partial y\partial x}$.

Examples 1. If $\alpha = t^2 i - tj + (2t+1)k$ and $\beta = (2t-3)i + j - tk$ then find $\frac{d}{dt}(\alpha\times\frac{d\beta}{dt})$ at $t=2$.

We have $\frac{d}{dt}(\alpha\times\frac{d\beta}{dt}) = \alpha\times\frac{d^2\beta}{dt^2} + \frac{d\alpha}{dt}\times\frac{d\beta}{dt}$.

Now $\alpha = t^2 i - tj + (2t+1)k = 4i - 2j + 5k$ at $t=2$

$\frac{d\alpha}{dt} = 2ti - j + 2k = 4i - j + 2k$ at $t=2$

$\beta = (2t-3)i + j - tk = i + j - 2k$ at $t=2$

$\frac{d\beta}{dt} = 2i - k$ at $t=2$, $\frac{d^2\beta}{dt^2} = 0$ at $t=2$

$$\therefore \frac{d}{dt}(\alpha\times\frac{d\beta}{dt}) = \begin{vmatrix} i & j & k \\ 4 & -2 & 5 \\ 0 & 0 & 0\end{vmatrix} + \begin{vmatrix} i & j & k \\ 4 & -1 & 2 \\ 2 & 0 & -1\end{vmatrix} = i + 8j + 2k.$$

2. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = -t-5$ where t is time. Find the components of its velocity and acceleration at time $t=1$ in the direction $i - 2j + 2k$.

Soln. velocity $\frac{dr}{dt} = 4ti + (2t-4)j - k = 4i - 2j - k$ at $t=1$

Unit vector in direction $i - 2j + 2k$ is $\frac{i-2j+2k}{\sqrt{1+4+4}} = \frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k$

Component of vel. in the given direction is

$$(4i - 2j - k)\cdot\left(\frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k\right) = 2.$$

Acceleration $\frac{d^2r}{dt^2} = 4i + 2j$

Component of Accⁿ in the given dirⁿ is $(4i+2j)\cdot\left(\frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k\right) = 0$

3. show that $A\cdot\frac{dA}{dt} = |A|\frac{d}{dt}|A|$

Soln. since $A\cdot A = |A|^2$, we have $\frac{d}{dt}(A\cdot A) = \frac{d}{dt}|A|^2$

$$\Rightarrow 2A\cdot\frac{dA}{dt} = 2|A|\frac{d|A|}{dt} \Rightarrow A\cdot\frac{dA}{dt} = |A|\frac{d}{dt}|A|.$$

Exercises: 1. If a and b are two vector functions of the scalar variable t, then P.T. $a \times \frac{d^2b}{dt^2} - \frac{d^2a}{dt^2}\times b = \frac{d}{dt}\left(a\times\frac{db}{dt} - \frac{da}{dt}\times b\right)$.

2. If $\frac{da}{dt} = r\times a$ and $\frac{db}{dt} = r\times b$ then show that $\frac{d}{dt}(a\times b) = r\times(a\times b)$, where r is a constant vector and a, b are vector functions of scalar variable t.

3. Let f depends on x, y, z, t where x, y, and z depend on t.
P.T. $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$.

4. Suppose a particle moves along a curve $r = (t^3 - 4t)i + (t^2+4t)j + (8t^2 - 3t^3)k$. Find the magnitudes of the tangential and normal components of its acceleration when $t=2$.

5. For the curve $r = (2a\cos t, 2a\sin t, bt^2)$, show that $[\dot{r}\,\ddot{r}\,\dddot{r}] = 8a^2bt$.

6. Find the unit normal to the surface
$r = a\cos u\sin v\, i + a\sin u\sin v\, j + a\cos u\, k$, $a>0$

7. Let $\frac{d^2A}{dt^2} = 6ti - 24t^2j + 4\sin t\, k$. Find A given that $A = 2i+j$ and $\frac{dA}{dt} = -i - 3k$ at $t=0$.

Gradient, Divergence, Curl

The vector differential operator del or nabla is defined as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

Gradient: Let $\phi(x, y, z)$ be a differentiable scalar function at each point (x, y, z) in a certain region of space. Then grad ϕ or $\nabla\phi$ is defined as

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

Directional derivative:

Consider a scalar function $\phi(x, y, z)$. Then the directional derivative of ϕ in the direction of a vector a is defined by $\nabla\phi \cdot \frac{a}{|a|}$ where $\frac{a}{|a|}$ is the unit vector in the dir^n of a.

Divergence: Let $V(x, y, z) = V_1 i + V_2 j + V_3 k$ be differentiable at each point (x, y, z) in a region of space. Then the divergence of V, written as $\nabla \cdot V$ or div V is defined as follows.

$$\nabla \cdot V = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(V_1 i + V_2 j + V_3 k)$$
$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Note that div V is a scalar.

Curl: Let $V(x, y, z) = V_1 i + V_2 j + V_3 k$ be a differentiable vector field. Then the curl or rotation of V, written as $\nabla \times V$, curl V, rot V is defined as follows:

$$\nabla \times V = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(V_1 i + V_2 j + V_3 k)$$
$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)i + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)j + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)k$$

Proposition 1. Suppose A and B are diff. vector functions, and ϕ, ψ are diff. scalar functions. Then the following laws hold:

(i) $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$, (ii) $\nabla\cdot(A+B) = \nabla\cdot A + \nabla\cdot B$

(iii) $\nabla \times (A+B) = \nabla\times A + \nabla\times B$ (iv) $\nabla\cdot(\phi A) = (\nabla\phi)\cdot A + \phi(\nabla\cdot A$

(v) $\nabla\times(\phi A) = \nabla\phi \times A + \phi(\nabla\times A)$, (vi) $\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$.

2. (i) $\nabla\cdot(\nabla\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

(ii) $\nabla\times(\nabla\phi) = 0$ (iii) $\nabla\cdot(\nabla\times A) = 0$

(iv) $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$.

Examples: 1. Find the directional derivative of $\phi = xy^2z + 4x^2z$ at $(-1,1,2)$ in the dir^n $(2i+j-2k)$.

We have $\nabla\phi = i(y^2z+8xz) + j(2xyz) + k(xy^2+4x^2)$

$= -14i - 4j + 3k$ at $(-1,1,2)$

The unit vector in the dir^n $(2i+j-2k)$ is $\hat{a} = \frac{2i+j-2k}{3}$.

$\therefore$ Directional derivative $= \nabla\phi\cdot\hat{a} = -\frac{38}{3}$.

2. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$,

Let $r = xi+yj+zk$ be the position vector of any point $P(x,y,z)$ on the surface. Then $dr = i\,dx + j\,dy + k\,dz$ lies in the tangent plane to the surface at P

Now $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$

$\Rightarrow \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)\cdot(i\,dx+j\,dy+k\,dz) = 0 \Rightarrow \nabla\phi\cdot dr = 0$

$\therefore \nabla\phi$ is perpendicular to dr and therefore to the surface.

3. If the vectors A and B are irrotational, then show that the vector $A\times B$ is solenoidal.

A vector a is said to be solenoidal if $\operatorname{div} a = 0$

Since A and B are irrotational, $\nabla\times A = \nabla\times B = 0$

Now $\operatorname{div}(A\times B) = \nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$

$= B\cdot 0 - A\cdot 0 = 0$.

4. Suppose $\nabla\times A = 0$, Evaluate $\nabla\cdot(A\times r)$.

Let $A = A_1 i + A_2 j + A_3 k$, $r = xi+yj+zk$. Then

$A\times r = (zA_2 - yA_3)i + (xA_3 - zA_1)j + (yA_1 - xA_2)k$.

and $\nabla\cdot(A\times r) = \frac{\partial}{\partial x}(zA_2 - yA_3) + \frac{\partial}{\partial y}(xA_3 - zA_1) + \frac{\partial}{\partial z}(yA_1 - xA_2)$

$= x\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + y\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + z\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$

$= r\cdot(\nabla\times A) = r\cdot \operatorname{curl} A = 0$ if $\nabla\times A = 0$.

5. Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$.

$\nabla^2\frac{1}{r} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$.

$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -x(x^2+y^2+z^2)^{-3/2}$, $\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$

Similarly, $\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}}$, $\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$

Then, by addition, we have $\nabla^2\left(\frac{1}{r}\right) = 0$.

Exercises:

1. Find the maximum value of the directional derivative of $\phi = x^2+z^2-y^2$ at $(1,3,2)$. Find also the direction in which it occurs.

2. Show that curl $u = 0$ if $u = (y^2+z^3)i + (2xy-5z)j + (3xz^2-5y)k$

3. Show that if $r = xi + yj + zk$ and, then
(i) curl $\frac{r}{|r|} = 0$ (ii) $\nabla \cdot \frac{r}{|r|} = \frac{2}{|r|}$, (iii) $\nabla \times \frac{r}{|r|^3} = 0$
(iv) curl grad $|r|^m = 0$.

4. Show that the vector $F = (2x-yz)i + (2y-zx)j + (2z-xy)k$ is irrotational. For this F, find a scalar function ϕ such that $F = \text{grad}\,\phi$.

5. If $\phi = (r\times a)\cdot(r\times b)$ then prove that
$\nabla\phi = b\times(r\times a) + a\times(r\times b)$.

6. Show that $\nabla f(x,y,z)$ is both irrotational and solenoidal vector if $f(x,y,z)$ satisfies $\nabla^2 f(x,y,z) = 0$.

Vector Integration

Ordinary Integrals of vector valued functions:-

If the vector function $R(t)$ of a scalar variable t be such that $a(t) = \frac{d}{dt}R(t)$
then $\int a(t)\,dt = R(u) + C$ where c is an arbitrary constant vector independent of t.

Let $a(t) = a_1(t)i + a_2(t)j + a_3(t)k$ be a vector function of a scalar variable t, where $a_1(t), a_2(t)$ and $a_3(t)$ are assumed continuous in a specific interval. Then

$$\int a(t)\,dt = i\int a_1(t)\,dt + j\int a_2(t)\,dt + k\int a_3(t)\,dt$$

is called an indefinite integral of $a(t)$.

A definite integral of $a(t)$ between the limits $t=a$ and $t=b$ can be written as

$$\int_a^b a(t)\,dt = [R(u)]_a^b = R(b) - R(a) \quad \text{if } R'(t) = a(t).$$

Line Integrals:- Let $r(t) = x(t)i + y(t)j + z(t)k$ be the position vector of $P(x,y,z)$ and suppose $r(t)$ defines a curve C joining points P_1 and P_2 where $t=t_1$ and $t=t_2$ resp. We assume that C is composed of a finite number of curves for each of which $r(t)$ has a continuous derivative.

Let $A(x,y,z) = A_1 i + A_2 j + A_3 k$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of A along C from P_1 to P_2 written as

$$\int_{P_1}^{P_2} A \cdot dr = \int_C A \cdot dr = \int_C A_1 dx + A_2 dy + A_3 dz .$$

is an example of a line integral.

If C is a closed curve, the integral around C is denoted as $\oint A \cdot dr = \oint A_1 dx + A_2 dy + A_3 dz$.

Thm: Suppose $A = \nabla\phi$ everywhere in a region R defined by $[a_1,a_2] \times [b_1,b_2] \times [c_1,c_2]$ and $\phi(x,y,z)$ is single valued and has continuous derivatives in R. Then

(i) $\int_{P_1}^{P_2} A \cdot dr$ is independent of the path C in R joining P_1 and P_2

(ii) $\oint_C A \cdot dr = 0$ around any closed curve in C in R

In such a case A is called a conservative vector field and ϕ is its scalar potential.

Surface Integral: Let S be a two-sided surface, where one side is considered arbitrarily +ve side (usually outer side if S is closed). A unit normal n to any point of the +ve side of S is called a positive or outward drawn unit normal.

Associated with the differential of surface area dS, a vector $d\vec{S}$ whose magnitude is dS and whose direction is that of $\vec{n}$. Then $d\vec{S} = \vec{n}\, dS$. The integral

$$\iint_S \vec{A} \cdot d\vec{S} = \iint_S \vec{A} \cdot \vec{n}\, dS$$ is called the flux of A over S

or a surface integral of A over S.

Other surface integrals are $\iint_S \phi\, dS$, $\iint_S \phi \vec{n}\, dS$, $\iint_S \vec{A} \times d\vec{S}$ where ϕ is a scalar function.

Volume Integral: Consider a closed surface in space enclosing a volume V. Then the following denote volume integrals or space integrals:

$$\iiint_V A\, dV \quad \text{and} \quad \iiint_V \phi\, dV$$

where A is a vector function and ϕ is a scalar function.

Examples 1. suppose $F = (5x^2+6y)i - (3x+2y^2)j + 2xz^2 K$ then evaluate $\int_C F \cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along the path C given by (i) $x = t,\ y = t^2,\ z = t^3$

(ii) the straight line joining $(0,0,0)$ to $(1,1,1)$.

we have $\int_C F \cdot dr = \int_C (5x^2+6y)dx - (3x+2y^2)dy + 2xz^2 dz$

(i) The points $(0,0,0)$ and $(1,1,1)$ refer to $t=0$ and $t=1$ along the curve $x=t,\ y=t^2,\ z=t^3$.

$$\therefore \int_C F \cdot dr = \int_0^1 \{(5t^2+6t)dt - (3t+2t^4)2t\,dt + 2t \cdot t^6\, 3t^2 dt\}$$

$$= \int_0^1 (5t^2 - 4t^5 + 6t^9)\,dt = \frac{8}{5}.$$

(ii) The eqn of the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$ say.

$$\int_C F \cdot dr = \int_0^1 (5t^2+6t)dt - (3t+2t^2)dt + 2t^3 dt$$

$$= \int_0^1 (2t^3 + 3t^2 + 3t)\,dt = \frac{2}{4} + 1 + \frac{3}{2} = 3.$$

2. Evaluate the surface integral $\iint_S (yz i + zx j + xy K) \cdot dS$ where S is the surface of the sphere $x^2+y^2+z^2 = 1$ in the first octant.

Here $F = yz i + zx j + xy K$ and $dS = i\,dy\,dz + j\,dz\,dx + K\,dx\,dy$

Thus the integral $= \iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy)$

Now $\iint_S yz\,dy\,dz = \int_{y=0}^{1} \int_{z=0}^{\sqrt{1-y^2}} yz\,dy\,dz = \int_0^1 y\,dy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-y^2}}$

$$= \frac{1}{2}\int_0^1 y(1-y^2)\,dy = \frac{1}{8}.$$

Similarly $\iint_S zx\,dz\,dx = \iint_S xy\,dx\,dy = \frac{1}{8}$.

Required integral $= 3 \times \frac{1}{8} = \frac{3}{8}$.

3. Let V be the closed region bounded by the surfaces $x=0,\ x=2;\ y=0,\ y=6;\ z=x^2,\ z=4$ and $F = yi + 2xj - zK$. Find $\iiint_V \nabla \times F\, dV$

we have $\nabla \times F = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2x & -z \end{vmatrix} = K.$

Therefore $\iiint_V \nabla \times F \, dV = K \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{4} dz\,dy\,dx$

$$= K \int_{x=0}^{2} dx \int_{y=0}^{6} (4-x^2)\,dy = 6K \int_0^2 (4-x^2)\,dx = 32K.$$

Exercises:

1. If $F = (5xy - 6x^2) i + (2y - 4x) j$ then evaluate $\int_C F \cdot dr$ where C is the curve in the xy plane given by $y = x^3$ from $(1,1)$ to $(2,8)$.

2. Evaluate $\int_C F \cdot dr$, where $F = (x^2 - 3y^2) i + (y^2 - 2x^2) j$ and C is the closed curve in the xy plane, given by $x = 3\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$, C is described in the anti-clockwise sense.

3. Given $A = (yz + 2x) i + xz j + (xy + 2z) K$. Evaluate $\int_C A \cdot dr$ along the curve $x^2 + y^2 = 1$, $z = 1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$.

4. Evaluate $\iint_S F \cdot n \, ds$, where $F = 6z i - 4j + yK$ and S is that part of the plane $2x + 6y + 3z = 10$ which is located in the first octant.

5. Suppose $A = 4xz i + xyz^2 j + 3zK$. Evaluate $\iint_S A \cdot n \, ds$ over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$.

6. Evaluate $\iint_R \sqrt{x^2+y^2} \, dx\,dy$ over the region R in the xy plane bounded by $x^2 + y^2 = 36$

7. Evaluate $\iiint_V (2x + y)\,dV$, where V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$.

8. Suppose $F = (2x^2 - 3z) i - 2xy j - 4xK$. Evaluate (a) $\iiint_V \nabla \cdot F \, dV$ and (b) $\iiint \nabla \times F \, dV$, where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$.

Divergence, Stokes' Theorem and Related Integral Theorems

Divergence Thm of Gauss:- Suppose V is the volume bounded by a closed surface S and A is a vector function with continuous derivatives. Then

$$\iiint_V \nabla\cdot A\,dV = \iint_S A\cdot n\,dS = \oiint_S A\cdot dS$$

where n is the outward drawn normal to S,

Stoke's Thm :- suppose S is an open, two-sided surface bdd by a closed, nonintersecting curve C (simple closed curve), and A is a continuously differentiable vector function. Then

$$\oint_C A\cdot dr = \iint_S (\nabla\times A)\cdot n\,dS$$

where C is traversed in the positive direction,

Green's thm in the Plane:- Suppose R is a closed region in the xy-plane bounded by a simple closed curve C, and suppose M and N are continuous functions of x and y having continuous derivatives in R, Then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in the positive (counter-clockwise) direction,

Examples

1. Use divergence thm to evaluate $\iint_S x^3 dydz + x^2y\,dzdx + x^2z\,dxdy$ where S is the closed surface consisting of the cylinder $x^2+y^2=4$ $(0\le z\le 3)$ and the circular disc $z=0$ and $z=3$ $(x^2+y^2\le 4)$.

The given surface integral is equivalent to the volume integral given by $\iiint_V \left[\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(x^2z)\right] dxdydz$

$$= \int_{z=0}^{3}\int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 5x^2\,dx\,dy\,dz = 20\int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{\sqrt{4-y^2}} x^2\,dx\,dy\,dz$$

$$= \frac{60}{3}\int_{y=0}^{2}(4-y^2)^{3/2}\,dy = \frac{60}{3}\times 3\pi = 60\pi.$$

2. Verify stoke's theorem for $F = (2x-y)i - yz^2 j - y^2 z K$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

The boundary C is given by the circle $x^2+y^2=1$.

$\therefore \int_C F.dr = \int_C (2x-y)dx$ [on C, $z=0$, $dz=0$]

$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta \, d\theta$ (Put $x=\cos\theta$, $y=\sin\theta$)

$= \int_0^{2\pi} \sin^2\theta \, d\theta = \pi$.

Now $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = K$.

On the surface of the sphere $x^2+y^2+z^2=1$, $n = xi+yj+zk$,

$\iint_S (\nabla \times F).n \, ds = \iint_S z \, ds = \iint_S z \frac{dx dy}{n.K} = \iint_S dx dy = \pi$

Thus stoke's thm is verified.

3. Verify Green's thm in a plane for $\oint_C \{(x^2+xy)dx + x dy\}$ where C is the curve enclosing the region bounded by $y=x^2$ and $y=x$.

The parabola $y=x^2$ and the line $y=x$ meets at $(0,0)$ and $(1,1)$.

Taking the given integral along $y=x^2$ from $(0,0)$ to $(1,1)$

$\int_0^1 (x^2+x^3)dx + x.2x^2 dx = \left[x^3 + \frac{x^4}{4}\right]_0^1 = \frac{5}{4}$.

Again, integrating along $y=x$ from $(1,1)$ to $(0,0)$,

$\int_1^0 2x^2 dx + x dx = \left[\frac{2}{3}x^3 + \frac{x^2}{2}\right]_0^1 = -\frac{7}{6}$.

Hence the line integral $= \frac{5}{4} - \frac{7}{6} = \frac{1}{12}$.

Now $\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$ [Since $M = x^2+xy$, $N = x$]

$= \iint_S (1-x) dx dy$

$= \int_{x=0}^1 \int_{y=x^2}^{x} (1-x) dy dx$

$= \int_0^1 (1-x)(x-x^2) dx = \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{12}$.

Thus, the theorem is verified.

Exercises: 1. Show that $\iint_S r \cdot ds = 3V$ where V is the volume enclosed by the closed surface S and r has its usual meaning.

2. Prove that $\iiint_V \nabla\phi\, dv = \iint_S \phi\, n\, ds$.
In particular show that $\iint_S n\, ds = 0$.

3. If $\operatorname{grad}\phi = F$ and $\nabla^2\phi = 0$, then show that for a closed surface S enclosing the volume V,
$$\iiint_V F^2\, dV = \iint_S \phi\, F \cdot n\, ds.$$

4. verify the divergence theorem for the vector function
$$F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)K$$
taken over the rectangular parallelopiped
$0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1.$

5. Use stoke's thm to show that $\int_C y\,dx + z\,dy + x\,dz = -2\sqrt{2}\pi a^2$
where C is given by $x^2+y^2+z^2 = 2ax + 2ay,\ x+y = 2a.$

6. Show that $\frac{1}{2}\oint_C (x\,dy - y\,dx)$ represents the area bounded by the closed curve C. Hence show that the area of the ellipse $x = a\cos\phi,\ y = b\sin\phi$ is πab.

7. verify Green's thm in a plane for
$$\oint_C \{(3x^2 - 6y^2)\,dx + (y - 3xy)\,dy\}$$
where C is the boundary of the region $x=0,\ y=0,\ x+y=1$.

8. verify Stoke's thm for the function $F = x^2 i + xy\,j$ integrated round the square in the plane $z=0$ whose sides are along the straight lines $x=0,\ y=0,\ x=a,\ y=a$.

9. verify the divergence thm of Gauss for $F = 2x^2 i + y j - z^2 K$ where S is the closed surface consisting of the closed surface of the cylinder $x^2+y^2=16$ between the planes $z=0$ and $z=2$ together with the circular ends of those planes.

10. Use Green's thm in a plane to show that
$$\oint_C \{(\cos x \sin y - xy)\,dx + \sin x \cos y\,dy\} = 0$$
where C is the circle $x^2+y^2=9$ in the xy plane described in the +ve sense

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